Ian Arriaga MacKenzie

Intoduction

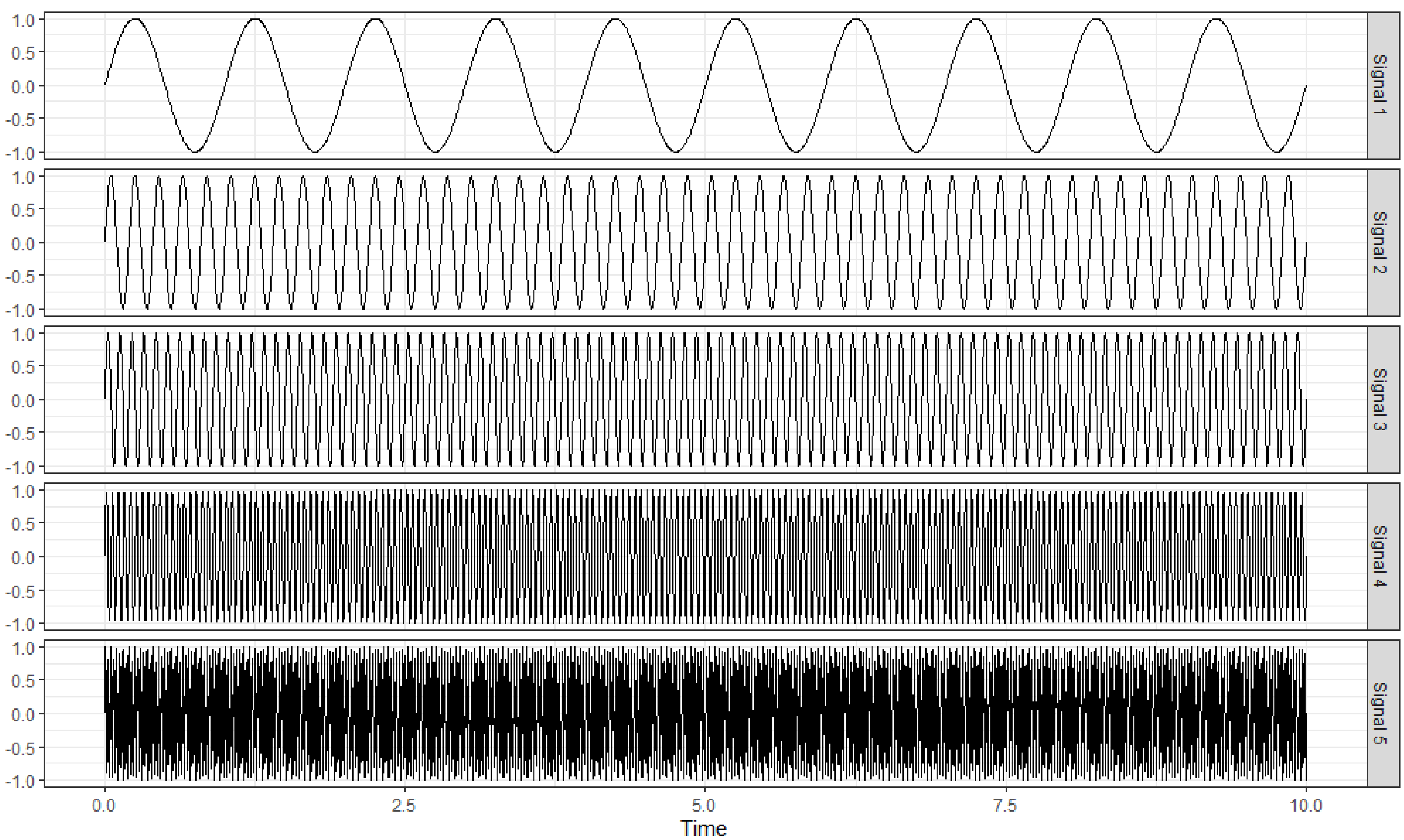
With time series data, we use the power spectrum to show the relative power of different frequencies of a recorded signal. Using Fourier analysis, and specifically the Fast Fourier Transform we can decompose a signal into either the specific frequencies or the distribution of frequencies it is made of. This form of algorithmic analysis of natural time series data is used in a wide range of fields, including physics, engineering, and biomedical applications.

Early applications were to sound waves and light spectra, but modern applications include telecommunications, geophysics, medical imaging, voice recognition, etc. There are many methods to estimate the power spectral density of a signal, however we focus on multitaper power spectral density (PSD) estimation. This form of PSD estimation uses multiple tapers to reduce the variance of the signal. These tapers are a series of orthogonal sine functions which can be used to efficiently reduce several forms of bias introduced in the process.

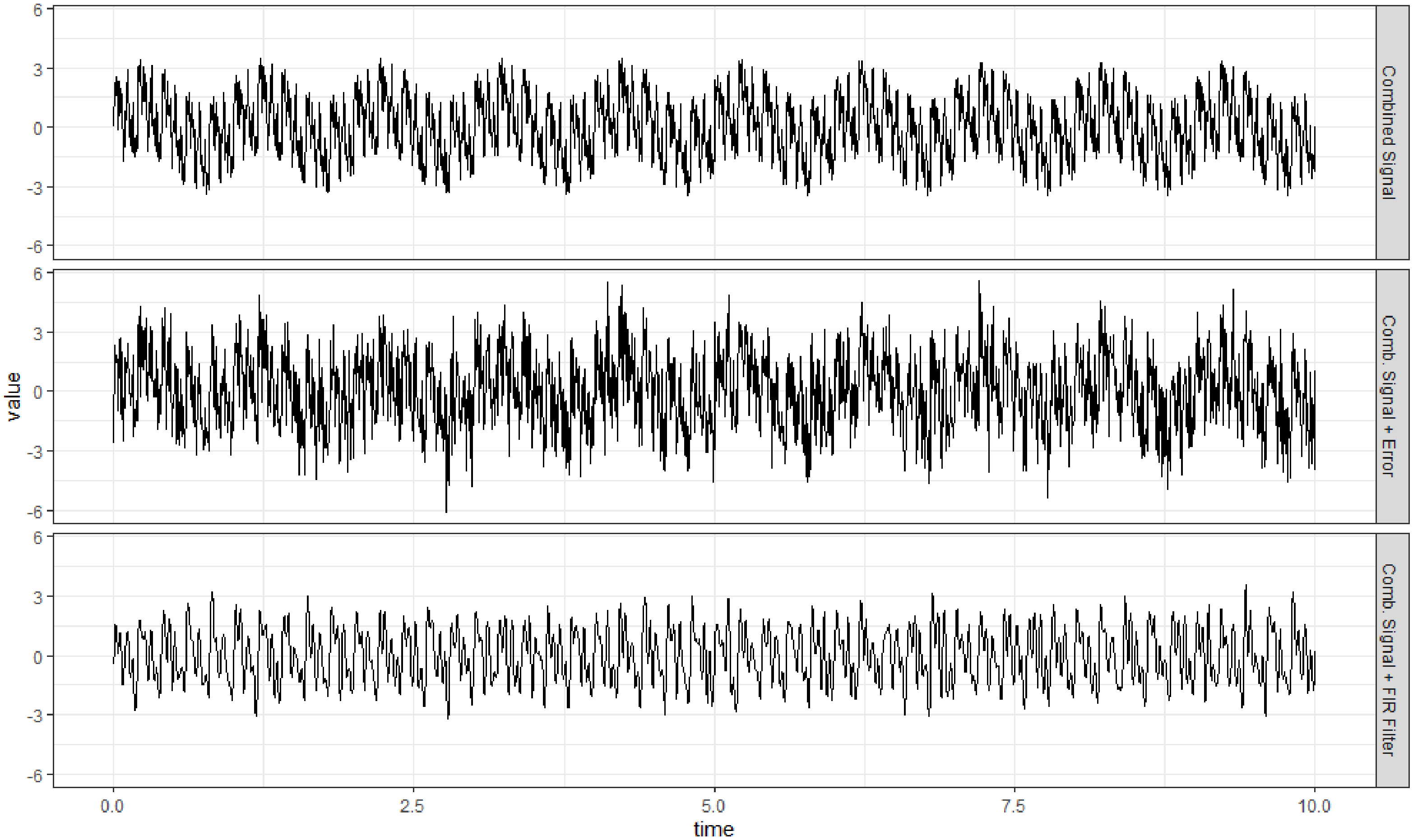
Case Study Description

To showcase how multitaper PSD estimation is used, we use two example datasets.

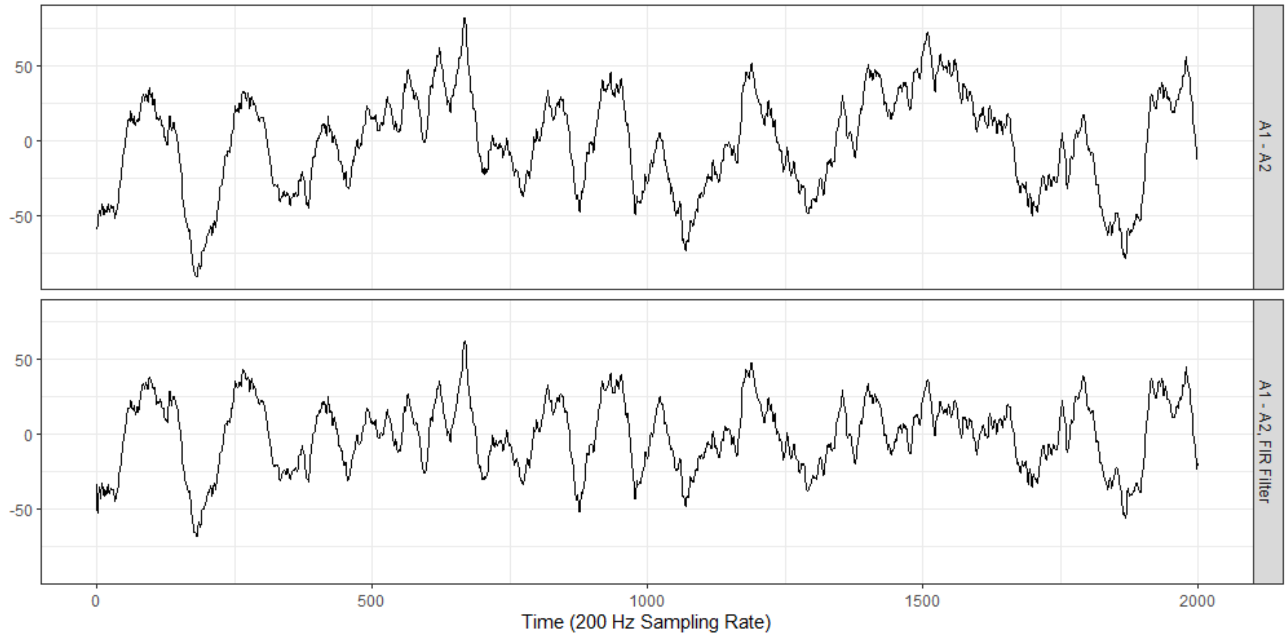
First, we use a simulated signal comprised of a mixture of five specific sinusoidal frequencies: 1 Hz, 5 Hz, 10 Hz, 20 Hz, and 45 Hz. The 1 Hz signal will repeat itself every second, the 5 Hz signal will repeat itself 5 times per second, and so forth. These individual signals at different frequencies are shown below, over a ten second interval:



The five signals are combined together to generate a mixed frequency signal, which we can use with PSD estimation to show signal decomposition. Further, we add simulated noise/error by adding a gaussian random sample (SD = 1) to our combined signal. We also use a band-pass (3 Hz – 30 Hz) finite impulse response (FIR) filter to show how we can target a specific range of frequencies for analysis (Helwig, 2018).



The second example dataset is an electroencephalogram (EEG) signal from a patient participating in a sleep study. This signal was measured from two leads placed near the ears of the patient (A1 to A2), and measure ionic current across the brain. A random 10 second sample of N2 stage sleep was taken from the patient, and both the raw and processed signal are shown below (0.5 Hz – 40 Hz bandpass FIR filter).



Model and Methods

Estimation of a spectrum has a history in physics and mathematics going back hundreds of years. Previously, spectral density was directly measured via instrumentation, however starting in the 1960's the advent of computing allowed for algorithmic decomposition of recorded signals and deeper inference in many aspects of science. Primarily, the ability to compute the discrete Fourier transform of a time series (using a fast Fourier transform algorithm) allowed for methods such as Welch's method to use time averaged windows in order to reduce the variance in estimates. (Welch, 1967).

In the decades since, there have been many methods developed to accurately decompose signals with minimal variance. Here, we focus on the method developed by Barbour et al. which uses multiple sine tapers to reduce forms of bias in estimating the power spectral density (Barbour et al., 2014). This method is based on the digital transform of the data:

$$\hat{S}_x(k\Delta f) = \frac{1}{T}\left|\sum_{j=0}^{N-1} x_j \exp\left(\frac{2\pi ijk}{N}\right)\right|^2, \quad k = 0,1,2,\ldots,N/2$$

where T is the record length of N samples x_j, and $\Delta f = 1/T$.

This base periodogram is susceptible to a form of bias known as spectral leakage, where the power of different frequencies are correlated to each other based on proximity. The multitaper method attempts to address this, however the choice of tapers introduces another form of bias known as curvature bias (the tapers over smooth the periodogram and power is reduced in the resulting PSD estimation). The tapers proposed by Barbour et al. are a series of sine functions which are computationally efficient and also reduce curvature bias, defined as follows:

$$\phi_k(t) = \sqrt{\frac{2}{T}} sin \frac{k\pi t}{T}, \qquad 0 \leq t \leq T, k = 1, 2, 3, \ldots$$

subject to

$$\int_0^T \prod_{k=1}^{M} \phi_k(t) dt = 0$$

These orthogonal functions ϕ_k are multiplied by the original input signal $x(t)$, which is then put into the digital Fourier transform, and the series of output periodograms are averaged together to estimate the power spectral density.

Other methods require a user to manually optimize the number of tapers, however this method is denoted as 'adaptive' PSD estimation in that Barbour shows the optimal number of tapers which reduce the mean square error is:

$$K_{opt}(f) = \left(\frac{12T^2 S(f)}{|S''(f)|}\right)^{2/5}$$

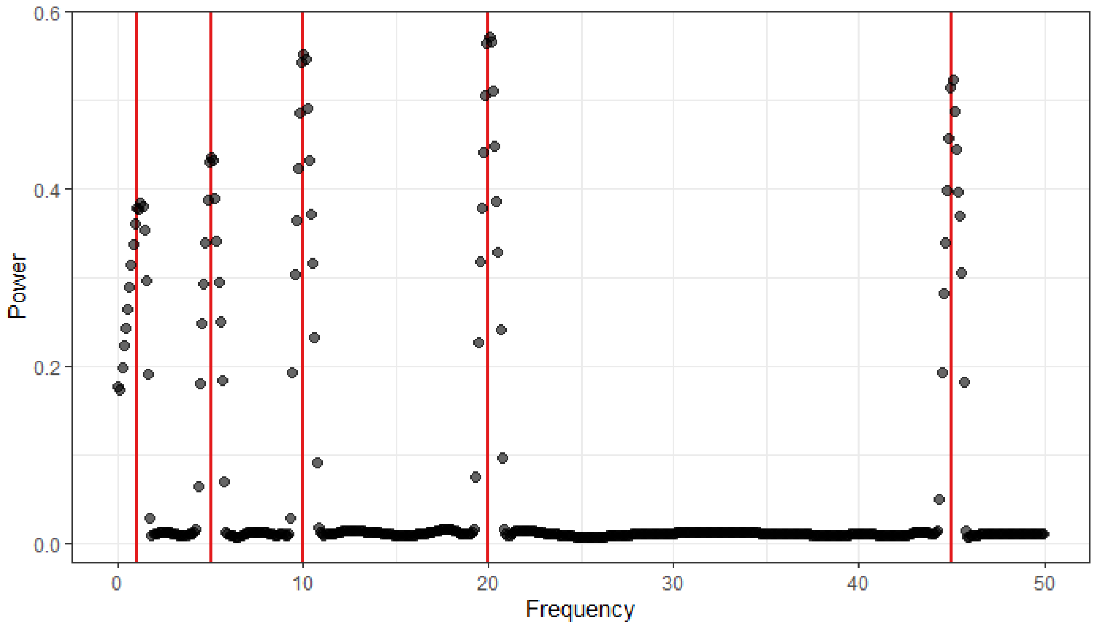
where $S(f)$ is the original power spectral density, and $S''(f)$ is the second derivative of PSD.

Therefor, the entire process is as follows:

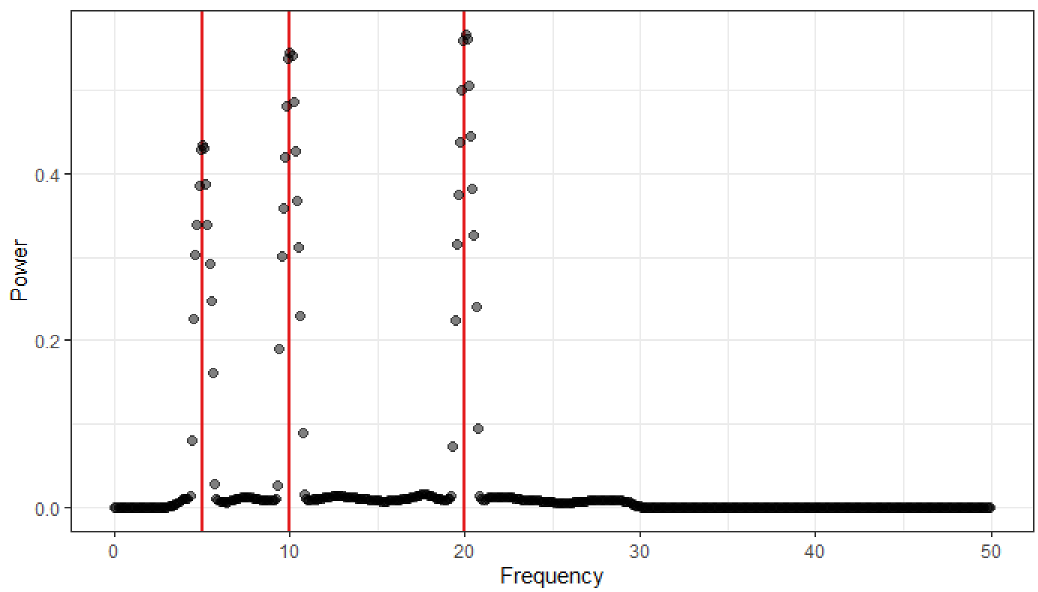
1. An original multitaper PSD is computed using a fixed number of tapers
2. The derivative and second derivative are computed, and used to generate the optimal number of tapers for the next iterate
3. A new multitaper PSD is computed using the updated number of tapers
4. Repeat until convergence, which happens when the change in number of tapers from K_i to K_{i+1} is one or less (generally 3 or 4 iterations)

Analysis and Results

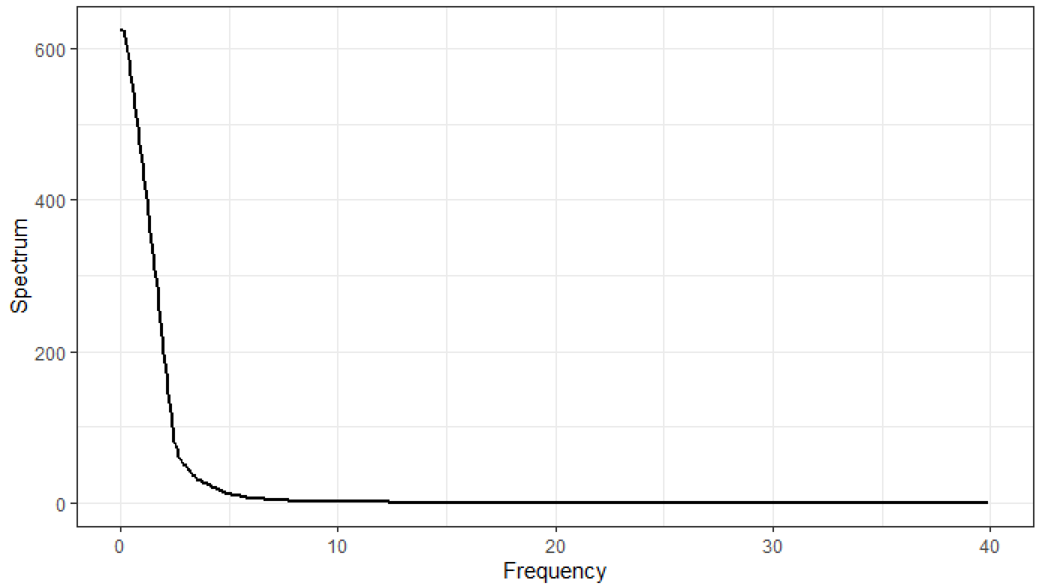
Using multitaper PSD estimation, we can decompose the two time series signals described in the case study above. Prior to applying the FIR filter, we can estimate the 5 frequencies which were used to create the simulated signal, even with gaussian noise added in.



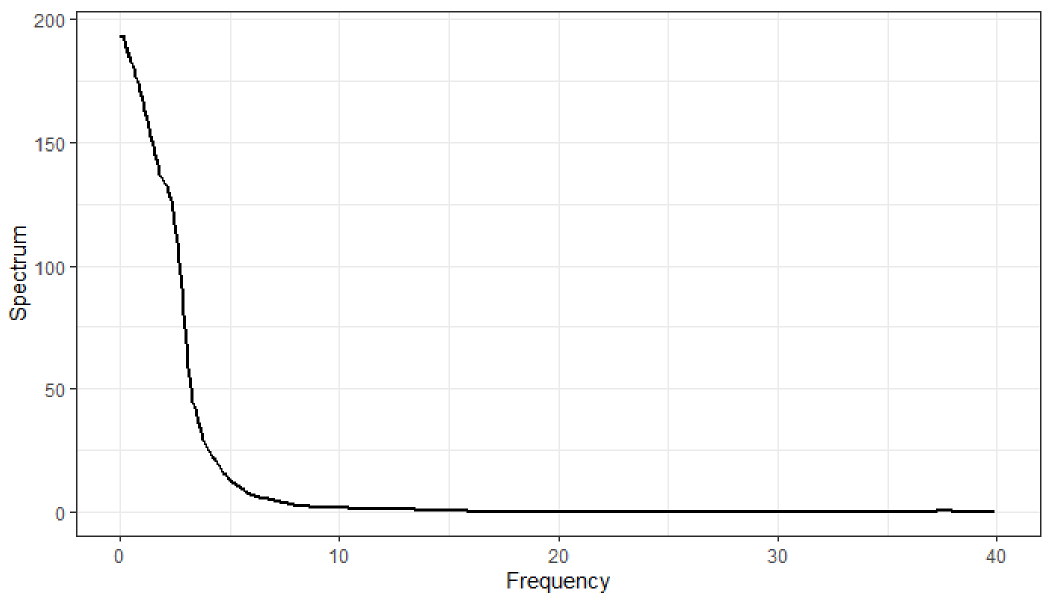
Next, we apply a 3 – 30 Hz band-pass FIR filter, to showcase the effect of isolating specific frequencies of interest. After applying the bandpass filter, we see that the power distribution in all frequencies outside the region of interest are near zero, and that we can still estimate the 3 frequencies present in the signal which were within the filter region.



We also apply multitaper PSD estimation to our EEG sample. In particular (because the patient was asleep) we focus on the lower frequencies (0.5 Hz – 40 Hz) as we expect there to be a large concentration of lower frequency brain waves (associated with sleeping).



Next, we apply a 0.5 Hz – 40 Hz band-pass FIR filter to the EEG sample. This allows us examine the lower frequency bandwidths more easily.



While this result may seem surprising at first, it does seem consistent with the data and highlights several features. Within EEG data, there are several defined frequency bandwidths which are associated with various states of consciousness. As this patient is asleep, we see high concentrations of Delta (0.5 – 4 Hz) and Theta (4 – 8 Hz) brain waves, with minimal higher brain waves. It also is visually consistent with the plotted EEG lead, as there are large sinusoidal waves in the reading.

However, for proper EEG analysis, several measurement leads across the head are used, of which multiple time points are sampled and averaged together so one 10 second anomaly does not affect the results. Here, we took a single 10 second sample, and therefor the results may not be representative.

Discussion

As we have shown, the method of adaptive multitaper PSD estimation developed by Barbour et al. is not only accurate, but also computationally efficient and fast. Running the algorithm on a personal computer takes seconds, and allows for quick resampling and averaging of the data. We also were accurately able to identify the 5 frequencies used in the mixed signal with gaussian error, both before and after a band-pass filter. PSD estimation has a wide range of uses, from electromagnetic inference as an example in the 'psd' R package, to EEG data as used in medical research.

References

Barbour et al. (2014). psd: Adaptive, sine multitaper power spectral density estimation for R. *Computers and Geosciences*.

Helwig, N. E. (2018, November 6). *Toolkit for Electroencephalography Data (eegkit)*. Retrieved from CRAN: https://cran.r-project.org/web/packages/eegkit/eegkit.pdf

Welch, P. D. (1967). The Use of Fast Fourier Transform for the Estimation of Power Spectra: A Method Based on Time Averaging Over Short, Modified Periodograms. *IEEE Transactions on Audio and Electroacoustics*, 70-73.